

THE INFLUENCE OF MAGNETIC STEPS ON BULK SUPERCONDUCTIVITY

WAFEE ASSAAD AND AYMAN KACHMAR

ABSTRACT. We study the distribution of bulk superconductivity in presence of an applied magnetic field, supposed to be a step function, modeled by the Ginzburg-Landau theory. Our results are valid for the minimizers of the two-dimensional Ginzburg-Landau functional with a large Ginzburg-Landau parameter and with an applied magnetic field of intensity comparable with the Ginzburg-Landau parameter.

1. INTRODUCTION AND MAIN RESULTS

Motivation. The Ginzburg-Landau functional successfully models the response of a (Type II) superconducting sample to an applied magnetic field. We focus on samples that occupy a long cylindrical domain and subjected to a magnetic field with direction parallel to the axis of the cylinder. This situation has been analyzed in many papers, see for instance the two monographs [8, 19]. However, in the literature, the focus was on uniform applied magnetic fields. Recently, attention has been shifted to non-uniform *smooth* magnetic fields in [2, 3, 13, 17]. Such magnetic fields may arise in the study of superconducting surfaces [7] or superconductors with applied electric currents [1].

In this paper, we consider the situation when the applied magnetic field is a step function. Such fields might occur in many situations (cf. [14]). In particular

- If a sample is separated into two parts, one can apply on one part a uniform magnetic field from above, and on the other part, a uniform magnetic field from below (see Figure 2).
- If a sample is not homogeneous, one can have a variable magnetic permeability. This leads to a magnetic step function (cf. [6]).

The functional. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and simply connected set and $B_0 : \Omega \rightarrow [-1, 1]$ be a measurable function. The Ginzburg-Landau functional in Ω is

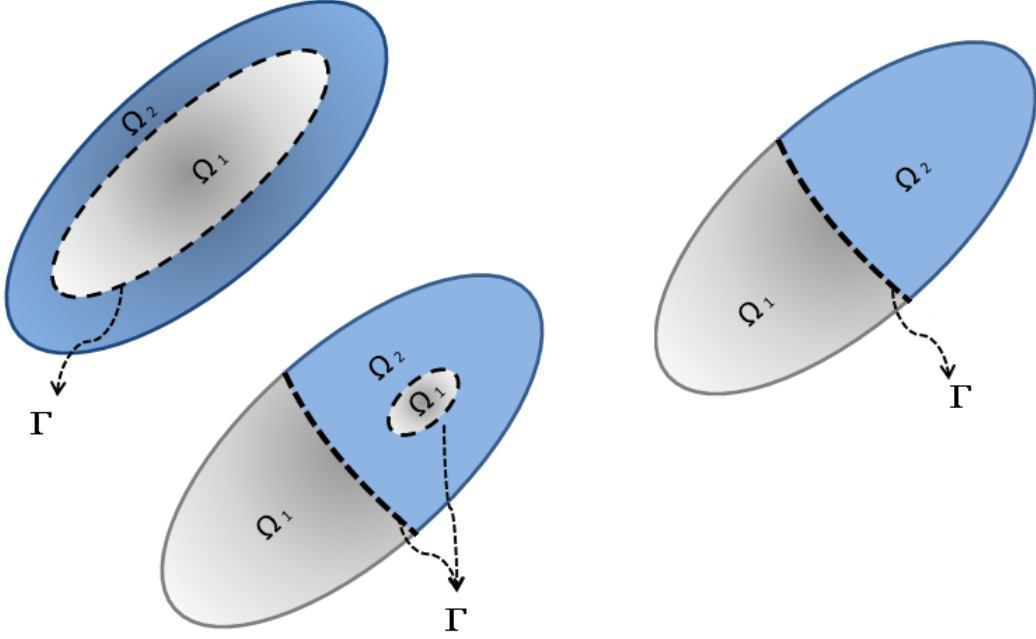
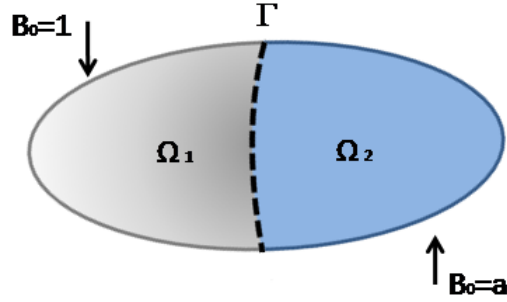
$$\mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) = \int_{\Omega} \left(|(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx + \kappa^2 H^2 \int_{\Omega} |\operatorname{curl} \mathbf{A} - B_0|^2 dx. \quad (1.1)$$

Here, $\kappa > 0$ is the Ginzburg-Landau parameter, a characteristic of the superconducting material, $H > 0$ is the intensity of the applied magnetic field, $\psi \in H^1(\Omega, \mathbb{C})$ and $\mathbf{A} = (A_1, A_2) \in H^1(\Omega, \mathbb{R}^2)$. In physics, the domain Ω is the cross section of the sample, the function B_0 is the applied magnetic field, the function ψ is the order parameter and the vector field \mathbf{A} is the magnetic potential. The configuration (ψ, \mathbf{A}) is interpreted as follows, $|\psi|^2$ measures the density of the superconducting electron pairs and $\operatorname{curl} \mathbf{A} = \partial_{x_1} A_2 - \partial_{x_2} A_1$ measures the induced magnetic field in the sample.

In this paper, we work under the following assumption on the function B_0 (these are illustrated in Figure 1):

Assumption 1.1.

- (1) $a \in [-1, 1] \setminus \{0\}$ is a given constant;
- (2) $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are two disjoint open sets;
- (3) $B_0 = \mathbf{1}_{\Omega_1} + a\mathbf{1}_{\Omega_2}$;
- (4) Ω_1 and Ω_2 have a finite number of connected components;

FIGURE 1. Schematic representations of the set Ω .FIGURE 2. Schematic representations of the set Ω subjected to a step magnetic field B_0 .

- (5) $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise smooth with (possibly) a finite number of corners ;
- (6) $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the union of a finite number of smooth curves ;
- (7) $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ and $\partial\Omega$ is of class C^4 ;
- (8) $\Gamma \cap \partial\Omega$ is either empty or finite ;
- (9) If $\Gamma \cap \partial\Omega$, then Γ intersects $\partial\Omega$ transversely, i.e. $\nu_{\partial\Omega} \times \nu_\Gamma \neq 0$, where $\nu_{\partial\Omega}$ and ν_Γ are respectively the unit normal vectors of $\partial\Omega$ and Γ .

We introduce the ground state energy

$$E_{\text{g.st}}(\kappa, H) = \inf\{\mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)\} \quad (1.2)$$

where

$$H_{\text{div}}^1(\Omega) = \{\mathbf{A} \in H^1(\Omega, \mathbb{R}^2) : \text{div} \mathbf{A} = 0 \text{ in } \Omega, \mathbf{A} \cdot \nu_{\partial\Omega} = 0 \text{ on } \partial\Omega\}. \quad (1.3)$$

The functional $\mathcal{E}_{\kappa, H}$ is invariant under the gauge transformations $(\psi, \mathbf{A}) \mapsto (e^{i\varphi\kappa H}\psi, \mathbf{A} + \nabla\varphi)$. This gauge invariance yields (cf. [8])

$$E_{\text{g.st}}(\kappa, H) = \inf\{\mathcal{E}_{\kappa, H}(\psi, \mathbf{A}) : (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}.$$

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ of $\mathcal{E}_{\kappa, H}$ are weak solutions of the following G-L equations:

$$\begin{cases} (\nabla - i\kappa H \mathbf{A})^2 \psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\ -\nabla^\perp(\text{curl } \mathbf{A} - B_0) = \frac{1}{\kappa H} \text{Im}(\bar{\psi}(\nabla - i\kappa H \mathbf{A})\psi) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H \mathbf{A}) = 0 & \text{on } \partial\Omega, \\ \text{curl } \mathbf{A} = B_0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Here $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ is the Hodge gradient.

Energy asymptotics and applications. The statement of our main results involves a continuous function $g : [0, \infty) \rightarrow [-\frac{1}{2}, 0]$ constructed in [20, 9]. The function g is increasing and satisfies $g(0) = -\frac{1}{2}$, $g(b) = 0$ for all $b \geq 1$, and $-\frac{1}{2} < g(b) < 0$ for all $b \in (0, 1)$.

Theorem 1.2. [Ground state energy asymptotics]

Let $\tau \in (\frac{3}{2}, 2)$ and $0 < c_1 < c_2$ be constants. Under Assumption 1.1, there exist constants $C > 0$ and $\kappa_0 > 0$ such that if

$$\kappa \geq \kappa_0 \quad \text{and} \quad c_1 \leq \frac{H}{\kappa} \leq c_2, \quad (1.5)$$

then the ground state energy in (1.2) satisfies

$$-C\kappa^\tau \leq E_{\text{g.st}}(\kappa, H) - \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \leq C\kappa^{3/2}. \quad (1.6)$$

Corollary 1.3. [L^4 -norm asymptotics of the order parameter]

Under the assumptions of Theorem 1.2, there exist constants $\kappa_0 > 0$ and $C > 0$ such that, if (1.5) holds, then:

(1) For every critical point (ψ, \mathbf{A}) of (1.1),

$$\int_{\Omega} |\psi|^4 dx + 2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \leq C\kappa^{\tau-2}.$$

(2) For every minimizer (ψ, \mathbf{A}) of (1.1),

$$\left| \int_{\Omega} |\psi|^4 dx + 2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \right| \leq C\kappa^{\tau-2},$$

and

$$\kappa^2 H^2 \int_{\Omega} |\text{curl } \mathbf{A} - B_0|^2 dx \leq C\kappa^\tau.$$

Theorem 1.4. [Distribution of the L^4 -norm of the order parameter]

Suppose that the assumptions of Theorem 1.2 are satisfied. Let $D \subset \Omega$ be an open set with a smooth boundary. There exist $\kappa_0 > 0$ and a function $\text{err} : (\kappa_0, \infty) \rightarrow \mathbb{R}_+$ such that $\lim_{\kappa \rightarrow \infty} \text{err}(\kappa) = 0$ and, if

- (1.5) holds;
- (ψ, \mathbf{A}) is a minimizer of (1.1);

then

$$\left| \int_D |\psi|^4 dx + 2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \right| \leq \text{err}(\kappa).$$

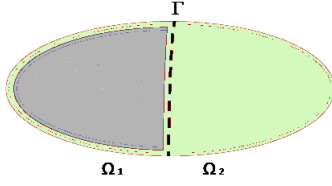


FIGURE 3. Schematic representation of Ω in the regime $\frac{H}{\kappa} = b$ where $1 < b < \frac{1}{|a|}$. The dark region is in a normal state, the bright region may carry superconductivity.

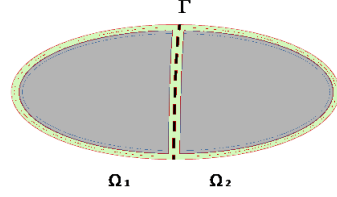


FIGURE 4. Schematic representation of Ω in the regime $\frac{H}{\kappa} = b$ where $b > \frac{1}{|a|}$. The dark region is in a normal state, the bright region may carry superconductivity.

Discussion of Theorem 1.4. The result in Theorem 1.4 displays the strength of the superconductivity in the bulk of Ω . We will use the following notation. Let $\omega \subset \mathbb{R}^2$ be an open set, (f_κ) be a sequence in $L^\infty(\omega)$, $\alpha \in \mathbb{C}$ and dx be the Lebesgue measure in \mathbb{R}^2 . By writing

$$f_\kappa dx \rightharpoonup \alpha dx \quad \text{in } \omega$$

we mean, for every ball $B \subset \omega$,

$$\int_B f_\kappa dx \rightarrow \alpha |B|.$$

Now we return back to the result in Theorem 1.4. Suppose that $H = b\kappa$ and $b \in (0, \infty)$ is a constant. Let us start by examining the case where $-1 < a < 1$. We observe that:

- (1) If $0 < b < 1$, then

$$|\psi|^4 dx \rightharpoonup -2g(b) dx \text{ in } \Omega_1 \quad \text{and} \quad |\psi|^4 dx \rightharpoonup -2g(b|a|) dx \text{ in } \Omega_2.$$

This means that the bulk of Ω carries superconductivity everywhere, but since $0 < |a| < 1$, $0 < -g(b) < -g(b|a|)$ and the strength of superconductivity in Ω_1 is smaller than that in Ω_2 .

- (2) If $1 \leq b < \frac{1}{|a|}$, then

$$|\psi|^4 dx \rightharpoonup 0 \text{ in } \Omega_1 \quad \text{and} \quad |\psi|^4 dx \rightharpoonup -2g(b|a|) dx \text{ in } \Omega_1,$$

with $g(b|a|) < 0$. In this regime, superconductivity disappears in the bulk of Ω_1 but persists in the bulk of Ω_2 . Theorem 1.5 below will sharpen this point by establishing that $|\psi|$ is exponentially small in the bulk of Ω_1 (see Figure 3). However, in light of the analysis in the book of Fournais-Helffer [8], the boundary of Ω_1 may carry superconductivity. This point deserves a detailed analysis.

- (3) If $b > \frac{1}{|a|}$, then superconductivity disappears in the bulk of Ω_1 and Ω_2 (see Figure 4).

However, one might find an interesting behavior near the critical value $b \sim \frac{1}{|a|}$. In the spirit of the analysis in [10], one expects to find superconductivity in the bulk of Ω_2 , but with a weak strength. This superconductivity is evenly distributed and decays as b gradually increases past the value $\frac{1}{|a|}$.

- (4) **[Break down of superconductivity ([12])]** If $b \gg \frac{1}{|a|}$, one expects that $\psi = 0$ and superconductivity is lost in the sample. To this end, the spectral analysis in [14] must be useful. In the spirit of the book [8], this regime is related to the analysis of the third critical field(s) where the transition to the purely normal state occurs.

The interesting case $a = -1$ is reminiscent of the situation of a *smooth* and sign-changing magnetic field analyzed in the paper by Helffer-Kachmar [13]. Note that Theorem 1.4 yields that superconductivity is evenly distributed in Ω_1 and Ω_2 as long as $0 < b < 1$. In the critical regime

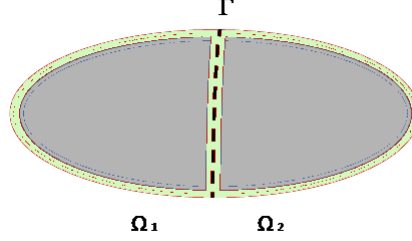


FIGURE 5. Schematic representation of the set Ω in the regime $\frac{H}{\kappa} = b$ where $a = -1$ and $b > 1$. The dark region is in a normal state, the bright region may carry superconductivity.

$b \sim 1$, one might find that superconductivity is distributed along the curve Γ that separates Ω_1 and Ω_2 , in the same spirit of the paper [13]. This behavior is illustrated in Figure 5.

Exponential decay in regions with larger magnetic intensity. Our last result establishes a regime for the strength of the magnetic field where the order parameter is exponentially small in the bulk of Ω_1 . The relevance of this theorem is that together with Theorem 1.4, display a regime of the intensity of the applied magnetic field such that $|\psi|^2$ is exponentially small in the bulk of Ω_1 while it is of the order $O(1)$ in Ω_2 .

Theorem 1.5. [Exponential decay of the order parameter]

Let $\lambda, \varepsilon, c_2 > 0$ be constants such that $0 < \varepsilon < \sqrt{\lambda}$ and $1 + \lambda < c_2$. There exist constants $C, \kappa_0 > 0$ such that, if

$$\kappa \geq \kappa_0, \quad (1 + \lambda)\kappa \leq H \leq c_2\kappa, \quad (\psi, \mathbf{A})_{\kappa, H} \text{ is a minimizer of (1.1),}$$

then

$$\begin{aligned} \int_{\Omega_1 \cap \{\text{dist}(x, \partial\Omega_1) \geq \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \frac{1}{\kappa H} |(\nabla - i\kappa H \mathbf{A})\psi|^2 \right) \exp \left(2\varepsilon \sqrt{\kappa H} \text{dist}(x, \partial\Omega_1) \right) dx \\ \leq C \int_{\Omega_1 \cap \{\text{dist}(x, \partial\Omega_1) \leq \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 dx. \end{aligned}$$

Unlike similar situations in [5, 11], we can not extend the result in Theorem 1.5 to *critical points* of the functional in (1.1). The technical reason behind this is as follows. A necessary ingredient in the proof given in [5, 11] is the following estimate of the magnetic energy

$$\|\text{curl } \mathbf{A} - B_0\|_{L^2(\Omega)} = o(\kappa^{-1}) \quad (\kappa \rightarrow \infty).$$

For critical configurations, we have the following estimate from [8, Lemma 10.3.2]

$$\|\text{curl } \mathbf{A} - B_0\|_{L^2(\Omega)} \leq C\kappa^{-1} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^4(\Omega)}.$$

To control the L^2 - and L^4 - norms of ψ , we use Theorem 1.4. But this will give that $\|\psi\|_{L^4(\Omega)} = o(1)$ only for $H \geq |a|^{-1}\kappa$, the condition necessary to get that $g(H\kappa^{-1}) = g(H\kappa^{-1}|a|) = 0$. This condition does not cover all the values of H in Theorem 1.5. As a substitute, we choose to control the magnetic energy by the estimate in Corollary 1.3, which is valid for *minimizing* configurations only.

Notation.

- The letter C denotes a positive constant whose value may change from line to line. Unless otherwise stated, the constant C depends on the function B_0 and the domain Ω , and independent of κ , H and the minimizers (ψ, \mathbf{A}) of the functional in (1.1).

- Given $\ell > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we denote by

$$Q_\ell(x) = \left(-\frac{\ell}{2} + x_1, \frac{\ell}{2} + x_1\right) \times \left(-\frac{\ell}{2} + x_2, \frac{\ell}{2} + x_2\right)$$

the square of side length ℓ centered at x .

- Let $a(\kappa)$ and $b(\kappa)$ be two positive functions, we write :
 - $a(\kappa) \ll b(\kappa)$, if $\frac{a(\kappa)}{b(\kappa)} \rightarrow 0$ as $\kappa \rightarrow \infty$.
 - $a(\kappa) \approx b(\kappa)$, if there exist constants κ_0, C_1 and C_2 such that for all $\kappa \geq \kappa_0$, $C_1 a(\kappa) \leq b(\kappa) \leq C_2 a(\kappa)$.
- The quantity $o(1)$ indicates a function of κ , defined by universal quantities, the domain Ω , given functions, etc and such that $|o(1)| \ll 1$. Any expression $o(1)$ is independent of the minimizer (ψ, \mathbf{A}) of (1.1). Similarly, $O(1)$ indicates a function of κ , bounded by a constant independent of the minimizers of (1.1).
- Let $n \in \mathbb{N}$, $p \in \mathbb{N}$. We use the following Sobolev spaces :

$$W^{n,p}(\Omega) := \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega), \text{ for all } |\alpha| \leq n\},$$

$$H^n(\Omega) := W^{n,2}(\Omega).$$

On the proofs and the organization of the paper. The results in this paper can be viewed as generalizations of those in [20] already proved for the case $B_0 = 1$. Theorem 1.5 is reminiscent of the exponential bounds in [5]. However, the proofs in this paper are simpler than those in [20] and contain new ingredients that we summarize below:

- We took advantage of all the available information regarding the limiting function $g(\cdot)$ proved in [9] and [3];
- We did not use the *a priori* elliptic estimates, e.g. the L^∞ -bound $\|(\nabla - i\kappa H \mathbf{A})\psi\|_\infty \leq C\kappa$. However, we used the simple energy bound $\|(\nabla - i\kappa H \mathbf{A})\psi\|_2 \leq C\kappa$ together with the regularity of the curl-div system (cf. Theorem 4.2). This method is already used for the three dimensional problem in [15];
- To prove Theorem 1.5, we did not establish *weak* decay estimates as done in [5].

The paper is divided into seven sections and two appendices. The first section is this introduction. Section 2 collects the needed properties of the limiting energy $g(\cdot)$. Section 3 establishes an upper bound of the ground state energy. Section 4 proves the necessary estimates on the critical points of the functional in (1.1). These estimates are used in Section 5 to establish a lower bound of the ground state energy. In Section 6, we finish the proof of Theorem 1.2, Corollary 1.3 and Theorem 1.4. Section 7 is devoted to the proof of Theorem 1.5. Finally, the appendices A and B collect standard results that are used throughout the paper.

2. THE LIMITING ENERGIES

Let $R > 0$ and $Q_R = (-R/2, R/2) \times (-R/2, R/2)$. We define the following Ginzburg-Landau energy with the constant magnetic field on $H^1(Q_R)$ by

$$G_{b,Q_R}^\sigma(u) = \int_{Q_R} \left(b |(\nabla - i\sigma \mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx. \quad (2.1)$$

Here $b \geq 0$, $\sigma \in \{-1, +1\}$ and \mathbf{A}_0 is the canonical magnetic potential

$$\mathbf{A}_0 = \frac{1}{2}(-x_2, x_1) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \quad (2.2)$$

which satisfies $\text{curl } \mathbf{A}_0 = 1$.

We introduce the two ground state energies

$$\begin{aligned} m_0(b, R, \sigma) &= \inf_{u \in H_0^1(Q_R)} G_{b,Q_R}^\sigma(u), \\ m(b, R, \sigma) &= \inf_{u \in H^1(Q_R)} G_{b,Q_R}^\sigma(u). \end{aligned} \quad (2.3)$$

Notice that $G_{b,Q_R}^{+1}(u) = G_{b,Q_R}^{-1}(\bar{u})$. As an immediate consequence, we observe that

$$\inf_{u \in \mathcal{V}} G_{b,Q_R}^{+1}(u) = \inf_{u \in \mathcal{V}} G_{b,Q_R}^{-1}(u) \quad \text{where} \quad \mathcal{V} \in \{H_0^1(Q_R), H^1(Q_R)\}, \quad (2.4)$$

and the values of $m_0(b, R, \sigma)$ and $m(b, R, \sigma)$ are independent of $\sigma \in \{-1, 1\}$. In the rest of the paper, we will denote these two values by $m_0(b, R)$ and $m(b, R)$ respectively, hence

$$m_0(b, R, \sigma) = m_0(b, R) \quad \text{and} \quad m(b, R, \sigma) = m(b, R) \quad (\sigma \in \{-1, 1\}). \quad (2.5)$$

We cite the following result from [3] (also see [9, 20]).

Theorem 2.1.

- (1) For all $b \geq 1$ and $R > 0$, we have $m_0(b, R) = 0$.
- (2) For all $b \in [0, \infty)$, there exists a constant $g(b) \leq 0$ such that

$$g(b) = \lim_{R \rightarrow \infty} \frac{m_0(b, R)}{R^2} = \lim_{R \rightarrow \infty} \frac{m(b, R)}{R^2} \quad \text{and} \quad g(0) = -\frac{1}{2}.$$

- (3) The function $[0, \infty) \ni b \mapsto g(b)$ is continuous, non-decreasing, concave and its range is the interval $[-\frac{1}{2}, 0]$.
- (4) There exists a constant $\gamma \in (0, \frac{1}{2})$ such that

$$\forall b \in [0, 1], \quad \gamma(b-1)^2 \leq |g(b)| \leq \frac{1}{2}(b-1)^2.$$

- (5) There exist constants C and R_0 such that, for all $R \geq R_0$ and $b \in [0, 1]$,

$$g(b) \leq \frac{m_0(b, R)}{R^2} \leq g(b) + \frac{C}{R} \quad \text{and} \quad g(b) - \frac{C}{R} \leq \frac{m(b, R)}{R^2} \leq g(b) + \frac{C}{R}.$$

3. ENERGY UPPER BOUND

The aim of this section is to prove :

Proposition 3.1. *Under the assumption of Theorem 1.2, there exist positive constants C and κ_0 such that if (1.5) holds, then the ground state energy $E_{g, \text{st}}(\kappa, H)$ in (1.2) satisfies*

$$E_{g, \text{st}}(\kappa, H) \leq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) dx + C\kappa^{3/2}.$$

Before writing the proof of Proposition 3.1, we introduce some notation. If $D \subset \Omega$ is an open set, we introduce the local energy of the configuration $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ in the domain $D \subset \Omega$ as follows

$$\begin{aligned} \mathcal{E}_0(\psi, \mathbf{A}; D) &= \int_D (|\nabla - i\kappa H \mathbf{A} \psi|^2 - \kappa^2 |\psi|^2 + \frac{1}{2} \kappa^2 |\psi|^4) dx, \\ \mathcal{E}(\psi, \mathbf{A}; D) &= \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_{\Omega} |\text{curl}(\mathbf{A} - \mathbf{F})|^2 dx. \end{aligned} \quad (3.1)$$

In Lemma A.1, we constructed a vector field \mathbf{F} satisfying

$$\mathbf{F} \in H_{\text{div}}^1(\Omega) \quad \text{and} \quad \text{curl} \mathbf{F} = B_0 \quad \text{in } \Omega. \quad (3.2)$$

Proof of Proposition 3.1.

Step 1. (Introducing a lattice of squares)

We introduce the small parameter

$$\ell = \kappa^{-1/2}. \quad (3.3)$$

Consider the lattice $L_{\ell} := \ell\mathbb{Z} \times \ell\mathbb{Z}$. Let

$$\mathcal{I}_{\ell}^1 = \{z \in L_{\ell} : \overline{Q_{\ell}(z)} \subset \Omega_1\}, \quad \mathcal{I}_{\ell}^2 = \{z \in L_{\ell} : \overline{Q_{\ell}(z)} \subset \Omega_2\} \quad \text{and} \quad \mathcal{I}_{\ell} = \mathcal{I}_{\ell}^1 \cup \mathcal{I}_{\ell}^2, \quad (3.4)$$

where $Q_{\ell}(z)$ denotes the square of center z and side-length ℓ . By Assumption 1.1, the number

$$N = \text{Card } \mathcal{I}_{\ell}$$

satisfies

$$|\Omega|\ell^{-2} - O(\ell^{-1}) \leq N \leq |\Omega|\ell^{-2} \quad (\ell \rightarrow 0_+). \quad (3.5)$$

Step 2. (Defining a trial state.)

For all $z \in \mathcal{I}_\ell$, let $\varphi_z \in C^2(\overline{Q_\ell(z)})$ be the function introduced in Lemma A.2 and

$$b_z = \frac{H}{\kappa} |B_0(z)|, \quad R_z = \ell \sqrt{\kappa H |B_0(z)|}, \quad \sigma_z = \frac{B_0(z)}{|B_0(z)|}. \quad (3.6)$$

The function φ_z satisfies

$$\mathbf{F}(x) = \nabla \varphi_z(x) + B_0(z) \mathbf{A}_0(x - z), \quad (x \in Q_\ell(z)). \quad (3.7)$$

We define the function $v \in H_0^1(\Omega)$ as follows,

$$v(x) = \begin{cases} e^{i\kappa H \varphi_z} u_{b_z, R_z, \sigma_z} \left(\frac{R_z}{\ell} (x - z) \right) & \text{if } x \in Q_\ell(z), \\ 0 & \text{if } x \in \Omega \setminus \Omega_\ell, \end{cases}$$

where

$$\Omega_\ell = \text{int} \left(\bigcup_{z \in \mathcal{I}_\ell} \overline{Q_\ell(z)} \right), \quad (3.8)$$

and $u_{b_z, R_z, \sigma_z} \in H_0^1(Q_R)$ is a minimizer of the functional in (2.1) (with $(b, R, \sigma) = (b_z, R_z, \sigma_z)$). In the sequel, we will omit the reference to (b_z, R_z, σ_z) in the notation u_{b_z, R_z, σ_z} and write simply

$$u_z = u_{b_z, R_z, \sigma_z}. \quad (3.9)$$

Step 3. (Energy of the trial state).

We compute the energy of the configuration (v, \mathbf{F}) . We have the obvious identities (cf. (3.1) and (3.2))

$$\begin{aligned} \mathcal{E}(v, \mathbf{F}; \Omega) &= \int_\Omega \left(|(\nabla - i\kappa H \mathbf{F})v|^2 - \kappa^2 |v|^2 + \frac{1}{2} \kappa^2 |v|^4 \right) dx \\ &= \sum_{z \in \mathcal{I}_\ell} \mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)). \end{aligned} \quad (3.10)$$

Using (3.7), we write

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) = \mathcal{E}_0(e^{-i\kappa H \varphi_z} v, \sigma_z |B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)).$$

By doing the change of variable $y = \frac{R_z}{\ell} (x - z)$, we get

$$\mathcal{E}_0(e^{-i\kappa H \varphi_z} v, \sigma_z |B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)) = \frac{\kappa}{H |B_0(z)|} \int_{Q_{R_z}} \left(b_z |(\nabla_y - i\sigma_z \mathbf{A}_0) u_z|^2 - |u_z|^2 + \frac{1}{2} |u_z|^4 \right) dy$$

where u_z is the function in (3.9) and (b_z, R_z, σ_z) is introduced in (3.6). By using (2.4), we get

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) = \mathcal{E}_0(e^{-i\kappa H \varphi_z} v, \sigma_z |B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)) = \frac{1}{b_z} m_0(b_z, R_z).$$

Since $\ell = \kappa^{-1/2}$ and $H \geq c_1 \kappa$, $R_z \geq 1$ (cf. (3.6)). We use Theorem (2.1) to write

$$m_0(b_z, R_z) \leq g(b_z) R_z^2 + C R_z.$$

Consequently,

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) \leq \ell^2 \kappa^2 g\left(\frac{H}{\kappa} |B_0(z)|\right) + C \ell \kappa. \quad (3.11)$$

We insert (3.11) into (3.10) to get

$$\begin{aligned} \mathcal{E}(v, \mathbf{F}; \Omega) &= \sum_{z \in \mathcal{I}_\ell} \left(\ell^2 \kappa^2 g\left(\frac{H}{\kappa} |B_0(z)|\right) + C \ell \kappa \right) \\ &\leq \kappa^2 \int_{\Omega_\ell} g\left(\frac{H}{\kappa} |B_0(x)|\right) dx + C \ell \kappa N, \end{aligned}$$

where $N = \text{Card } \mathcal{I}_\ell$. Now, using (3.5) and the fact that $-\frac{1}{2} \leq g(\cdot) \leq 0$, we get

$$\mathcal{E}(v, \mathbf{F}; \Omega) \leq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) dx + \frac{1}{2} |\Omega \setminus \Omega_\ell| \kappa^2 + C \frac{\kappa}{\ell}.$$

To finish the proof of Proposition 3.1, we use that $E_{\text{g.st}}(\kappa, H) \leq \mathcal{E}(v, \mathbf{F}; \Omega)$, $\ell = \kappa^{-1/2}$ and that the regularity of $\partial\Omega$ together with Assumption 1.1 yields

$$|\Omega \setminus \Omega_\ell| = O(\ell) \text{ as } \ell \rightarrow 0_+. \quad (3.12)$$

□

4. A PRIORI ESTIMATES

In the derivation of a lower bound of the energy in (1.1) various error terms arise. These terms are controlled by the estimates that we derive in this section.

In Proposition 4.1, we state a celebrated estimate of the order parameter:

Proposition 4.1. *If $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ is a weak solution to (1.4), then*

$$\|\psi\|_{L^\infty} \leq 1.$$

A detailed proof of Proposition 4.1 can be found in [8, Proposition 10.3.1]. It only needs the assumption that $B_0 \in L^2(\Omega)$.

Proposition 4.1 is used in the next theorem to give a priori estimates on the solutions of the Ginzburg-Landau equations (1.4).

Theorem 4.2. *Let $0 < c_1 < c_2$ and $\alpha \in (0, 1)$ be constants. Suppose that the conditions in Assumption 1.1 hold.*

There exist two constants $\kappa_0 > 0$ and $C > 0$ such that, if (1.5) holds and $(\psi, \mathbf{A}) \in H^1(\Omega) \times H_{\text{div}}^1(\Omega)$ is a solution of (1.4), then

- (1) $\|(\nabla - i\kappa H \mathbf{A})\psi\|_{L^2(\Omega)} \leq C\kappa;$
- (2) $\|\text{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \frac{C}{\kappa};$
- (3) $\mathbf{A} - \mathbf{F} \in H^2(\Omega)$ and $\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq \frac{C}{\kappa};$
- (4) $\mathbf{A} - \mathbf{F} \in C^{0,\alpha}(\overline{\Omega})$ and $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leq \frac{C}{\kappa}.$

Proof. The inequalities in items (1) and (2) in Theorem 4.2 follow from Lemma 10.3.2 in [8].

Now we prove item (3) in Theorem 4.2. Let $a = \mathbf{A} - \mathbf{F} \in H_{\text{div}}^1(\Omega)$. By (1.4), we know that $a \in H^1(\Omega)$ and $\text{curl} a \in H_0^1(\Omega)$. Using Lemma B.1 and the second equation in (1.4), we get $a \in H^2(\Omega)$ and,

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq C \|\nabla(\text{curl}(\mathbf{A} - \mathbf{F}))\|_{L^2(\Omega)} = \frac{C}{\kappa H} \|\overline{\psi}(\nabla - i\kappa H \mathbf{A})\psi\|_{L^2(\Omega)}.$$

Using the bound $H \geq c_1 \kappa$, Proposition 4.1 and the estimate in item (2) in Theorem 4.2, we get the estimate in item (3) above.

Finally, the conclusion in item (4) in Theorem 4.2 is simple a consequence of the conclusion in item (3) and the Sobolev embedding of $H^2(\Omega)$ in $C^{0,\alpha}(\overline{\Omega})$. □

Remark 4.3. In Theorem 4.2, the constant C depends on α only in item (4). Later in this paper, a fixed value of α is chosen. For this reason, we simply denote this constant by C instead of $C(\alpha)$.

Remark 4.4. In Theorem 4.2, Assumption 1.1 is used in the derivation of items (3) and (4) only. In fact, Assumption 1.1 ensures that the domains Ω_1 and Ω_2 satisfy the *cone condition*, which in turn allows us to use the Sobolev embedding theorems (cf. e.g. the proof of Lemma B.1).

5. ENERGY LOWER BOUND

The aim of this section is to establish a lower bound for the ground state energy in (1.2). This will be done in two steps (cf. Lemma 5.1 and Proposition 5.2 below). As a consequence of the results in this section, we will be able to finish the proof of Theorem 1.2 and Corollary 1.3.

Recall that $Q_\ell(x_0)$ denotes the square of center x_0 and side length ℓ . In the statements of Lemma 5.1, Proposition 5.2 and Theorem 5.3, we will use the functional \mathcal{E}_0 in (3.1).

Lemma 5.1. *Let $\alpha \in (0, 1)$ and $0 < c_1 < c_2$ be constants. There exist positive constants C and κ_0 such that, if*

- (1.5) holds;
- $0 < \delta < 1$, $0 < \ell < 1$, $x_0 \in \Omega$;
- $\overline{Q_\ell(x_0)} \subset \Omega_i$ for some $i \in \{1, 2\}$;
- $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a critical point of (1.1);
- $h \in C^1(\overline{\Omega})$, $\|h\|_\infty \leq 1$;

then the following inequality holds

$$\begin{aligned} \mathcal{E}_0(h\psi, \mathbf{A}; Q_\ell(x_0)) &\geq (1 - \delta) \mathcal{E}_0(e^{-i\kappa H \eta} h\psi, \sigma_\ell(x_0) |B_0(x_0)| \mathbf{A}_0(x - x_0); Q_\ell(x_0)) \\ &\quad - C(\delta^{-1} \ell^{2\alpha+2} \kappa^2 + \delta \ell^2 \kappa^2), \end{aligned}$$

for some function $\eta \in C^2(\overline{Q_\ell(x_0)})$.

Proof. Let $\phi_{x_0}(x) = (\mathbf{A}(x_0) - \mathbf{F}(x_0)) \cdot x$. Using Theorem 4.2, we get for all $x \in Q_\ell(x_0)$

$$\begin{aligned} |\mathbf{A}(x) - \nabla \phi_{x_0}(x) - \mathbf{F}(x)| &\leq \|(\mathbf{A} - \mathbf{F})\|_{C^{0,\alpha}(\overline{\Omega_i})} |x - x_0|^\alpha \\ &\leq \frac{C}{\kappa} \ell^\alpha. \end{aligned} \tag{5.1}$$

Define $\eta = \varphi_{x_0} + \phi_{x_0}$ where φ_{x_0} is the function introduced in Lemma A.2 and satisfying

$$\mathbf{F}(x) = \nabla \varphi_{x_0}(x) + \sigma_{x_0} |B_0(x_0)| \mathbf{A}_0(x - x_0), \quad \sigma_{x_0} = \frac{B_0(x_0)}{|B_0(x_0)|}, \quad x \in Q_\ell(x_0).$$

Let

$$u = e^{-i\kappa H \eta} h\psi. \tag{5.2}$$

Using the gauge invariance, the Cauchy-Schwartz inequality and (5.1), we write

$$|(\nabla - i\kappa H \mathbf{A})h\psi|^2 \geq (1 - \delta) |(\nabla - i\kappa H(\sigma_{x_0} |B_0(x_0)| \mathbf{A}_0(x - x_0)))u|^2 - C\delta^{-1} \ell^{2\alpha} \kappa^2 h^2 |\psi|^2.$$

Now, by recalling the definition of u and by using the estimates $\|\psi\|_\infty \leq 1$ and $\|h\|_\infty \leq 1$, we deduce the following lower bound of $\mathcal{E}_0(h\psi, \mathbf{A}; Q_\ell(x_0))$,

$$\mathcal{E}_0(h\psi, \mathbf{A}; Q_\ell(x_0)) \geq (1 - \delta) \mathcal{E}_0(u, \sigma_{x_0} |B_0(x_0)| \mathbf{A}_0(x - x_0); Q_\ell(x_0)) - C(\delta^{-1} \ell^{2\alpha+2} \kappa^2 + \delta \ell^2 \kappa^2).$$

□

Proposition 5.2. *Under the assumptions in Lemma 5.1, it holds*

$$\mathcal{E}_0(h\psi, \mathbf{A}; Q_\ell(x_0)) \geq \ell^2 \kappa^2 g\left(\frac{H}{\kappa} |B_0(x_0)|\right) - C(\ell^{2\alpha+1} \kappa^2 + \ell \kappa + \ell^3 \kappa^2). \tag{5.3}$$

Proof. Let

$$b = \frac{H}{\kappa} |B_0(x_0)|, \quad R = \ell \sqrt{\kappa H |B_0(x_0)|}, \quad \sigma_{x_0} = \frac{B_0(x_0)}{|B_0(x_0)|},$$

and define the rescaled function $v(x) = u(\frac{\ell}{R}x + x_0)$ for all $x \in Q_R = (-R/2, R/2)^2$, where the function u is defined in (5.2). The change of variable $y = \frac{R}{\ell}(x - x_0)$ yields

$$\mathcal{E}_0(u, \sigma_{x_0} |B_0(x_0)| \mathbf{A}_0(x - x_0); Q_\ell(x_0)) = \frac{1}{b} G_{b, Q_R}^{\sigma_{x_0}}(v). \tag{5.4}$$

Since $v \in H^1(Q_R)$, then by (2.3), (2.5) and Theorem 2.1,

$$\begin{aligned} \mathcal{E}_0(u, \sigma_{x_0} | B_0(x_0) | \mathbf{A}_0(x - x_0); Q_\ell(x_0)) &\geq \frac{1}{b} m(b, R) \\ &\geq \frac{1}{b} (g(b)R^2 - CR). \end{aligned}$$

Inserting this into the estimate in Lemma 5.1 and taking $\delta = \ell$, we finish the proof of Proposition 5.2. \square

In the next theorem, we establish a lower bound of the local energy in an open subset D of Ω . Note that, in the particular case $h = 1$ and $D = \Omega$, Theorem 5.3 yields the lower bound in Theorem 1.2.

Theorem 5.3. *Let $\tau \in (\frac{3}{2}, 2)$ and $0 < c_1 < c_2$ be constants. Suppose that $D \subset \Omega$ is an open set with a smooth boundary. There exist constants $C > 0$ and $\kappa_0 > 0$ such that, if*

- (1.5) holds;
- $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H_{\text{div}}^1(\Omega)$ is a critical point of (1.1);
- $h \in C^1(\overline{\Omega})$, $\|h\|_\infty \leq 1$;

then the following inequality holds

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) \geq \kappa^2 \int_D g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx - C\kappa^\tau.$$

Proof. Let $\ell \in (0, 1)$ and $\alpha \in (0, 1)$ be defined as follows

$$\alpha = \frac{1}{2(\tau - 1)} \quad \text{and} \quad \ell = \kappa^{1-\tau} \quad (5.5)$$

In particular, we observe that ℓ is a function of κ such that $\ell \ll 1$ as $\kappa \rightarrow \infty$. Consider the lattice $L_\ell = \ell\mathbb{Z} \times \ell\mathbb{Z}$ as in Proposition 3.1. Let

$$\begin{aligned} \mathcal{I}_\ell^1(D) &= \{z \in L_\ell : \overline{Q_\ell(z)} \subset D \cap \Omega_1\} \\ \mathcal{I}_\ell^2(D) &= \{z \in L_\ell : \overline{Q_\ell(z)} \subset D \cap \Omega_2\} \\ \mathcal{I}_\ell(D) &= \mathcal{I}_\ell^1(D) \cup \mathcal{I}_\ell^2(D), \end{aligned} \quad (5.6)$$

$$N_1 = \text{Card}(\mathcal{I}_\ell^1(D)), \quad N_2 = \text{Card}(\mathcal{I}_\ell^2(D)), \quad N = N_1 + N_2 = \text{Card}(\mathcal{I}_\ell(D)),$$

and

$$D_\ell = \text{int} \left(\bigcup_{z \in \mathcal{I}_\ell(D)} \overline{Q_\ell(z)} \right).$$

Notice that

$$|D|\ell^{-2} - \mathcal{O}(\ell^{-1}) \leq N \leq |D|\ell^{-2}, \quad (5.7)$$

and

$$|D \setminus D_\ell| = \mathcal{O}(\ell) \quad (\ell \rightarrow 0_+). \quad (5.8)$$

Recall b_z and R_z defined in (3.6),

$$b_z = \frac{H}{\kappa} |B_0(z)| \quad \text{and} \quad R_z = \ell \sqrt{\kappa H |B_0(z)|},$$

Let (ψ, \mathbf{A}) be a minimizer of (1.1). We decompose $\mathcal{E}_0(h\psi, \mathbf{A}; D)$ as follows :

$$\mathcal{E}_0(h\psi, \mathbf{A}; D) = \mathcal{E}_0(h\psi, \mathbf{A}; D_\ell) + \mathcal{E}_0(h\psi, \mathbf{A}; D \setminus D_\ell). \quad (5.9)$$

Using (5.8), $|\psi| \leq 1$, $|h| \leq 1$ and Item (1) in Theorem 4.2, we get

$$\begin{aligned} |\mathcal{E}_0(h\psi, \mathbf{A}; D \setminus D_\ell)| &\leq \int_{D \setminus D_\ell} (|(\nabla - i\kappa H \mathbf{A})h\psi|^2 + \kappa^2 h^2 |\psi|^2 + \frac{\kappa^2}{2} h^4 |\psi|^4) dx \\ &\leq C\kappa^2 |D \setminus D_\ell| \\ &\leq C\kappa^2 \ell. \end{aligned} \quad (5.10)$$

On the other hand, we have the obvious decomposition of the energy in D_ℓ as follows

$$\mathcal{E}_0(h\psi, \mathbf{A}; D_\ell) = \sum_{z \in \mathcal{I}_\ell(D)} \mathcal{E}_0(h\psi, \mathbf{A}; Q_\ell(z)).$$

Using Proposition 5.2 and the estimate in (5.7), we get

$$\mathcal{E}_0(h\psi, \mathbf{A}; D_\ell) \geq \ell^2 \kappa^2 \sum_{z \in \mathcal{I}_\ell(D)} g\left(\frac{H}{\kappa} |B_0(z)|\right) - C(\ell^{2\alpha-1} \kappa^2 + \ell^{-1} \kappa + \ell \kappa^2).$$

Since $g(\cdot) \leq 0$, $D_\ell \subset D$ and B_0 is a step function,

$$\ell^2 \sum_{z \in \mathcal{I}_\ell(D)} g\left(\frac{H}{\kappa} |B_0(z)|\right) = \int_{D_\ell} g\left(\frac{H}{\kappa} |B_0(x)|\right) dx \geq \int_D g\left(\frac{H}{\kappa} |B_0(x)|\right) dx.$$

Consequently, we get the following lower bound

$$\mathcal{E}_0(h\psi, \mathbf{A}; D_\ell) \geq \kappa^2 \int_D g\left(\frac{H}{\kappa} |B_0(x)|\right) dx - C(\ell^{2\alpha-1} \kappa^2 + \ell^{-1} \kappa + \ell \kappa^2). \quad (5.11)$$

Now, the choice of ℓ and α in (5.5) allows us to infer from (5.10) and (5.11)

$$\mathcal{E}_0(h\psi, \mathbf{A}; D \setminus D_\ell) \geq -C\kappa^\tau \quad \text{and} \quad \mathcal{E}_0(h\psi, \mathbf{A}; D_\ell) \geq \kappa^2 \int_D g\left(\frac{H}{\kappa} |B_0(x)|\right) dx - C\kappa^\tau.$$

Inserting these two estimates in (5.9) finishes the proof of Theorem 5.3. \square

6. PROOF OF THE MAIN THEOREMS: ENERGY AND L^4 -NORM ASYMPTOTICS

This section is devoted to the proof of Theorem 1.2, Corollary 1.3 and Theorem 1.4.

Proof of Theorem 1.2. The proof follows by collecting the results in Proposition 3.1 and Theorem 5.3. \square

Proof of Corollary 1.3. Let (ψ, \mathbf{A}) be a critical point of (1.1). In light of (1.4), the function ψ satisfies

$$-(\nabla - i\kappa H \mathbf{A})^2 \psi = \kappa^2 (1 - |\psi|^2) \psi \text{ in } \Omega. \quad (6.1)$$

Multiply both sides of (6.1) by $\bar{\psi}$, integrate by parts over Ω and use the boundary condition in (1.4) to obtain

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) = -\frac{1}{2} \kappa^2 \int_\Omega |\psi|^4 dx. \quad (6.2)$$

Here \mathcal{E}_0 is the function in (3.1). Using Theorem 5.3 with $h = 1$ and $D = \Omega$, we get

$$\int_\Omega |\psi|^4 \leq -2 \int_\Omega g\left(\frac{H}{\kappa} |B_0(x)|\right) dx + C\kappa^{\tau-2}. \quad (6.3)$$

Now, we assume in addition that (ψ, \mathbf{A}) is a minimizer of (1.1). We will determine a lower bound of $\|\psi\|_4$ matching with the upper bound in (6.3). To that end, we observe that

$$\kappa^2 H^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2 dx + \mathcal{E}_0(\psi, \mathbf{A}; \Omega) = E_{\text{g.st}}(\kappa, H). \quad (6.4)$$

This yields that

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \leq E_{\text{g.st}}(\kappa, H).$$

The upper bound in Theorem 3.1 gives

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \leq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\kappa^{3/2}.$$

Inserting this into (6.2), we obtain

$$\int_{\Omega} |\psi|^4 \geq -2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx - C\kappa^{1/2}. \quad (6.5)$$

It remains to prove the estimate of the magnetic energy. In fact, we get by (6.4), Proposition 3.1 and Theorem 5.3

$$\kappa^2 H^2 \int_{\Omega} |\operatorname{curl} \mathbf{A} - B_0|^2 dx \leq E_{\text{g.st}}(\kappa, H) - \mathcal{E}_0(\psi, \mathbf{A}; \Omega) \leq C\kappa^{\tau}.$$

□

Proof of Theorem 1.4. The proof we give does not use *a priori* elliptic estimates, hence, it diverges from the one used by Sandier-Serfaty in [20]. As a substitute of the elliptic estimates, we use the *a priori* estimates obtained by an energy argument in Theorem 4.2, and a trick that we borrow from a paper by Helffer-Kachmar in [13].

Note that the particular case $D = \Omega$ is handled by Corollary 1.3. Now we establish Theorem 1.4 in the general case when D is an open subset of Ω with a smooth boundary. The proof is decomposed into two steps:

Upper bound:

For $\ell \in (0, 1)$, we define the two sets

$$\begin{aligned} D_{\ell} &= \{x \in D : \operatorname{dist}(x, \partial D) \geq \ell\}, \\ (\Omega \setminus D)_{\ell} &= \{x \in \Omega \setminus D : \operatorname{dist}(x, \partial D) \geq \ell\}. \end{aligned} \quad (6.6)$$

Since the boundary of D is smooth, we get

$$|D \setminus D_{\ell}| = \mathcal{O}(\ell) \quad (\ell \rightarrow 0_+). \quad (6.7)$$

However, the boundary of $\Omega \setminus D$ might have a finite number of cusps. But we still have,

$$\epsilon(\ell) := \left| (\Omega \setminus D) \setminus (\Omega \setminus D)_{\ell} \right| \rightarrow 0 \quad \text{as } \ell \rightarrow 0_+. \quad (6.8)$$

Let $\chi_{\ell} \in C_c^{\infty}(\mathbb{R}^2)$ be a cut-off function satisfying

$$0 \leq \chi_{\ell} \leq 1 \text{ in } \mathbb{R}^2, \quad \operatorname{supp} \chi_{\ell} \subset D, \quad \chi_{\ell} = 1 \text{ in } D_{\ell} \quad \text{and} \quad |\nabla \chi_{\ell}| \leq \frac{C}{\ell} \text{ in } \mathbb{R}^2, \quad (6.9)$$

where C is a universal constant.

Multiply both sides of the equation

$$-(\nabla - i\kappa H \mathbf{A})^2 \psi = \kappa^2(1 - |\psi|^2)\psi$$

by $\chi_{\ell}^2 \bar{\psi}$, integrate by parts over D , then use (6.9), (6.7) and $\|\psi\|_{\infty} \leq 1$ to get

$$\begin{aligned} \int_D \left(|(\nabla - i\kappa H \mathbf{A})\chi_{\ell}\psi|^2 - \kappa^2 \chi_{\ell}^2 |\psi|^2 + \kappa^2 \chi_{\ell}^2 |\psi|^4 \right) dx &= \int_D |\nabla \chi_{\ell}|^2 |\psi|^2 dx \\ &\leq C\ell^{-1}. \end{aligned} \quad (6.10)$$

Notice that $\chi_{\ell}^4 \leq \chi_{\ell}^2 \leq 1$. We infer from (6.10)

$$\mathcal{E}_0(\chi_{\ell}\psi, \mathbf{A}; D) \leq -\frac{1}{2}\kappa^2 \int_D \chi_{\ell}^4 |\psi|^4 dx + C\ell^{-1}. \quad (6.11)$$

Again, using (6.9), (6.7) and $\|\psi\|_\infty \leq 1$, we obtain

$$\begin{aligned} \int_D |\psi|^4 dx &= \int_D \chi_\ell^4 |\psi|^4 dx + \int_D (1 - \chi_\ell^4) |\psi|^4 dx \\ &\leq \int_D \chi_\ell^4 |\psi|^4 dx + C\ell. \end{aligned}$$

Consequently, in light of (6.11), we write,

$$\mathcal{E}_0(\chi_\ell \psi, \mathbf{A}; D) \leq -\frac{1}{2} \kappa^2 \int_D |\psi|^4 dx + C(\ell \kappa^2 + \ell^{-1}).$$

Now Theorem 5.3 with $h = \chi_\ell$ yields

$$\int_D |\psi|^4 dx \leq -2 \int_D g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + C(\ell + \ell^{-1} \kappa^{-2} + \kappa^{\tau-2}).$$

Choosing $\ell = \kappa^{-\frac{1}{2}}$, we get

$$\int_D |\psi|^4 dx \leq -2 \int_D g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + C\kappa^{\tau-2}. \quad (6.12)$$

In a similar fashion, by using the alternative property in (6.8), we get,

$$\begin{aligned} \int_{\Omega \setminus D} |\psi|^4 dx &\leq -2 \int_{\Omega \setminus D} g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + C\kappa^{\tau-2} + C\epsilon(\ell) \\ &= -2 \int_{\Omega \setminus D} g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + o(1). \end{aligned} \quad (6.13)$$

Lower bound:

Since ∂D is smooth, $|\partial D| = 0$ and the following simple decomposition holds

$$\int_D |\psi|^4 dx = \int_\Omega |\psi|^4 dx - \int_{\Omega \setminus D} |\psi|^4 dx.$$

By (6.13), we have

$$\int_{\Omega \setminus D} |\psi|^4 dx \leq -2 \int_{\Omega \setminus D} g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + o(1).$$

Using (6.5) and (6.12), we deduce that

$$\int_D |\psi|^4 dx = \int_D g\left(\frac{H}{\kappa} |B_0(x_0)|\right) dx + o(1).$$

□

7. EXPONENTIAL DECAY AND PROOF OF THEOREM 1.5

A key ingredient in the proof of Theorem 1.5 is

Lemma 7.1. *Let $\alpha \in (0, 1/4)$ be a constant. Under the assumptions in Theorem 1.5, there exist two constants $\kappa_0 > 0$ and $C > 0$ such that for all $\kappa \geq \kappa_0$ and $\phi \in C_c^\infty(\Omega_1)$,*

$$\|(\nabla - i\kappa H \mathbf{A})\phi\|_{L^2(\Omega_1)}^2 \geq \kappa H (1 - C\kappa^{-\alpha}) \|\phi\|_{L^2(\Omega_1)}^2.$$

Proof. Let $\phi \in C_c^\infty(\Omega_1)$. We write the following well known spectral estimate (cf. e.g. [8, Lem. 1.4.1])

$$\int_{\Omega_1} |(\nabla - i\kappa H \mathbf{A})\phi|^2 dx \geq \kappa H \int_{\Omega_1} \text{curl } \mathbf{A} |\phi|^2 dx.$$

Using the simple decomposition $\operatorname{curl} \mathbf{A} = (\operatorname{curl} \mathbf{A} - B_0) + B_0$, $B_0 = 1$ in Ω_1 , the Cauchy-Schwartz inequality and the estimate of $\|\operatorname{curl} \mathbf{A} - B_0\|_2$ in Corollary 1.3 (with $\tau = 2 - 2\alpha$), we get,

$$\begin{aligned} \int_{\Omega_1} |(\nabla - i\kappa H \mathbf{A})\phi|^2 dx &\geq \kappa H \int_{\Omega_1} |\phi|^2 dx + \kappa H \int_{\Omega_1} (\operatorname{curl} \mathbf{A} - B_0) |\phi|^2 dx \\ &\geq \kappa H \int_{\Omega_1} |\phi|^2 dx - \kappa H \|\operatorname{curl} \mathbf{A} - B_0\|_{L^2(\Omega)} \|\phi\|_{L^4(\Omega_1)}^2 \\ &\geq \kappa H \left[\|\phi\|_{L^2(\Omega_1)}^2 - C\kappa^{-1-\alpha} \|\phi\|_{L^4(\Omega_1)}^2 \right]. \end{aligned}$$

Now, we use the Sobolev embedding of $H^1(\Omega_1)$ into $L^4(\Omega_1)$ and the diamagnetic inequality to get, for all $\eta \in (0, 1)$ (cf. [11, Eq. (4.14)]),

$$\begin{aligned} \|\phi\|_{L^4(\Omega_1)}^2 &\leq C_{\text{Sob}} \left(\eta \|\nabla |\phi|\|_{L^2(\Omega_1)}^2 + \eta^{-1} \|\phi\|_{L^2(\Omega_1)}^2 \right) \\ &\leq C_{\text{Sob}} \left(\eta \|(\nabla - i\kappa H \mathbf{A})\phi\|_{L^2(\Omega_1)}^2 + \eta^{-1} \|\phi\|_{L^2(\Omega_1)}^2 \right). \end{aligned}$$

To finish the proof, we select $\eta = \kappa^{-1}$. □

Now, the proof of Theorem 1.5 follows similarly as [11, Thm. 4.1].

Proof of Theorem 1.5. Define the distance function t on Ω_1 as follows :

$$t(x) = \operatorname{dist}(x, \partial\Omega_1).$$

Let $\tilde{\chi} \in C^\infty(\mathbb{R})$ be a function satisfying

$$\tilde{\chi} = 0 \text{ on } (-\infty, \frac{1}{2}], \quad \tilde{\chi} = 1 \text{ on } [1, \infty) \text{ and } |\nabla \tilde{\chi}| \leq m,$$

where m is a universal constant.

Define the two functions χ and f on Ω_1 as follows :

$$\begin{aligned} \chi(x) &= \tilde{\chi}(\sqrt{\kappa H} t(x)), \\ f(x) &= \chi(x) \exp(\varepsilon \sqrt{\kappa H} t(x)), \end{aligned}$$

where ε is a positive number whose value will be fixed later.

Using the first equation of (1.4), we multiply both sides by $f^2 \bar{\psi}$ and we integrate by parts over Ω_1 , it results

$$\begin{aligned} \int_{\Omega_1} \left(|(\nabla - i\kappa H \mathbf{A})f\psi|^2 - |\nabla f|^2 |\psi|^2 \right) dx &= \kappa^2 \int_{\Omega_1} (|\psi|^2 - |\psi|^4) f^2 dx \\ &\leq \kappa^2 \int_{\Omega_1} |\psi|^2 f^2 dx. \end{aligned} \tag{7.1}$$

We combine the conclusions in (7.1) and Lemma 7.1 to get

$$\begin{aligned} \int_{\Omega_1} |\nabla f|^2 |\psi|^2 dx &\geq (\kappa H (1 - C\kappa^{-\alpha}) - \kappa^2) \|f\psi\|_{L^2(\Omega_1)}^2 \\ &\geq (\lambda - C(1 + \lambda)\kappa^{-\alpha}) \kappa^2 \|f\psi\|_{L^2(\Omega_1)}^2. \end{aligned} \tag{7.2}$$

Now, we estimate the term on the right side of (7.2) as follows

$$\int_{\Omega_1} |\nabla f|^2 |\psi|^2 dx \leq \varepsilon^2 \kappa H \|f\psi\|_{L^2(\Omega_1)}^2 + C\kappa H \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \leq 1\}} |\psi|^2 dx$$

Inserting (7.3) into (7.2) and dividing by κ^2 yields

$$(\lambda - C(1 + \lambda)\kappa^{-\alpha} - \varepsilon^2) \|f\psi\|_{L^2(\Omega_1)}^2 \leq C \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \leq 1\}} |\psi|^2 dx.$$

We choose the constant ε such that $0 < \varepsilon < \sqrt{\lambda}$. That way, we get for κ sufficiently large,

$$\int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \geq 1\}} |f\psi|^2 dx \leq \tilde{C} \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \leq 1\}} |\psi|^2 dx, \quad (7.3)$$

where \tilde{C} is a constant. Inserting (7.2) and (7.3) into (7.1) finishes the proof of Theorem 1.5. \square

We conclude this paper by Theorem 7.2 below, whose proof is similar to that of Theorem 1.5.

Theorem 7.2. [Exponential decay in Ω_2]

Let $\lambda, c_2 > 0$ be two constants such that $\frac{1}{|a|} + \lambda < c_2$. There exist constants $C, \varepsilon, \kappa_0 > 0$ such that, if

$$\kappa \geq \kappa_0, \quad \left(\frac{1}{|a|} + \lambda \right) \kappa \leq H \leq c_2 \kappa, \quad (\psi, \mathbf{A})_{\kappa, H} \text{ is a minimizer of (1.1),}$$

then

$$\begin{aligned} \int_{\Omega_2 \cap \{\text{dist}(x, \partial\Omega_2) \geq \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \frac{1}{\kappa H} |(\nabla - i\kappa H \mathbf{A})\psi|^2 \right) \exp \left(2\varepsilon \sqrt{\kappa H} \text{dist}(x, \partial\Omega_2) \right) dx \\ \leq C \int_{\Omega_2 \cap \{\text{dist}(x, \partial\Omega_1) \leq \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 dx. \end{aligned}$$

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APPENDIX A. GAUGE TRANSFORMATION

Lemma A.1. Suppose that Ω satisfies the conditions in Assumption 1.1. Let $B_0 \in L^2(\Omega)$. There exists a unique vector field $\mathbf{F} \in H_{\text{div}}^1(\Omega)$ such that

$$\text{curl } \mathbf{F} = B_0.$$

Furthermore, F is in $C^\infty(\Omega_1) \cup C^\infty(\Omega_2)$ and in $H^2(\Omega_1) \cup H^2(\Omega_2)$.

Proof. Let $f \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution of $-\Delta f = B_0$ in Ω (cf. [8]).

The vector field $\mathbf{F} = (\partial_{x_2} f, -\partial_{x_1} f) \in H_{\text{div}}^1(\Omega)$ and satisfies $\text{curl } \mathbf{F} = B_0$.

Since B_0 is constant in Ω_i for $i \in \{1, 2\}$, f the solution of $-\Delta f = B_0$ becomes in $C^\infty(\Omega_i)$. This yields that \mathbf{F} is in $C^\infty(\Omega_i)$, $i \in \{1, 2\}$.

That $\mathbf{F} \in H^2(\Omega_i)$ follows from the regularity of the solution of the equation $-\Delta f = B_0$ with Dirichlet condition on $\partial\Omega$ and continuity conditions on $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (cf. [16, Thm. B.1]). The uniqueness of \mathbf{F} is a consequence of the estimate (cf. [8, Prop. D.2.1] or [18, Appendix 1])

$$\forall a \in H_{\text{div}}^1(\Omega), \quad \|a\|_{H^1(\Omega)} \leq C_* \|\text{curl } a\|_{L^2(\Omega)}, \quad (\text{A.1})$$

where $C_* > 0$ is a universal constant. \square

Lemma A.2. Let $i \in \{1, 2\}$, $\ell \in (0, 1)$, $x_0 \in \Omega$ and $Q_\ell(x_0) \subset \Omega_i$. There exists a function $\varphi_{x_0} \in C^2(Q_\ell(x_0))$ such that the magnetic potential \mathbf{F} defined in Lemma A.1 satisfies

$$\mathbf{F}(x) = B_0(x_0)A_0(x - x_0) + \nabla \varphi_{x_0}(x), \quad (x \in Q_\ell(x_0))$$

where B_0 is the function defined in 1.1 and \mathbf{A}_0 is the magnetic potential introduced in (2.2).

Proof. By the definition of \mathbf{F} and \mathbf{A}_0 we have for all $x \in Q_\ell(x_0)$,

$$\text{curl } \mathbf{F}(x) = B_0(x) \text{curl } \mathbf{A}_0(x) \quad \text{in } Q_\ell(x_0).$$

Since $Q_\ell(x_0)$ is simply connected and B_0 is constant in $Q_\ell(x_0)$, we get the existence of the function φ_{x_0} . \square

APPENDIX B. curl-div ELLIPTIC ESTIMATE

Lemma B.1. *Suppose that Ω is simply connected and satisfies the conditions in Assumption 1.1. There exists a constant $C > 0$ such that, if $a \in H_{\text{div}}^1(\Omega)$ and $\text{curl } a \in H_0^1(\Omega)$, then $a \in H^2(\Omega)$ and the following inequality holds*

$$\|a\|_{H^2(\Omega)} \leq C \|\nabla(\text{curl } a)\|_{L^2(\Omega)}.$$

Proof. Let $f \in H_0^1(\Omega) \cap H^2(\Omega)$ be the unique solution of the Dirichlet problem $-\Delta f = \text{curl } a$ in Ω . By the inequality in (A.1), we get that $a = \nabla^\perp f = (\partial_{x_2} f, -\partial_{x_1} f)$.

Since $f = 0$ on $\partial\Omega$ and $\text{curl } a \in H^1(\Omega)$, we have by the elliptic estimates

$$\|f\|_{H^2(\Omega)} \leq C (\|\text{curl } a\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}),$$

$f \in H^3(\Omega)$ and

$$\begin{aligned} \|f\|_{H^3(\Omega)} &\leq C (\|\text{curl } a\|_{H^1(\Omega)} + \|f\|_{H^2(\Omega)}) \\ &\leq C (\|\text{curl } a\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}). \end{aligned}$$

This proves that $a \in H^2(\Omega)$. Now, since $f = 0$ and $\text{curl } a = 0$ on $\partial\Omega$, we get by the Poincaré inequality

$$\|f\|_{H^3(\Omega)} \leq C (\|\nabla(\text{curl } a)\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}).$$

To finish the proof of Lemma B.1, we simply observe that, since $a = \nabla^\perp f$, $\|\nabla f\|_{L^2(\Omega)} = \|a\|_{L^2(\Omega)}$ and consequently

$$\begin{aligned} \|a\|_{L^2(\Omega)} &\leq C_* \|\text{curl } a\|_{L^2(\Omega)} && [\text{By (A.1)}] \\ &\leq C \|\nabla(\text{curl } a)\|_{L^2(\Omega)} && [\text{By the Poincaré inequality}]. \end{aligned}$$

□

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LEBANESE INTERNATIONAL UNIVERSITY, DEPARTMENT OF MATHEMATICS, SAIDA CAMPUS, LEBANON
E-mail address: Wafaa_assaad@hotmail.com

LEBANESE UNIVERSITY, DEPARTMENT OF MATHEMATICS, HADAT, LEBANON
E-mail address: ayman.kashmar@gmail.com